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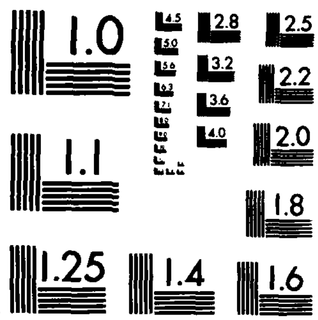

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ON PAIRS OF POSITIVE SOLUTIONS FOR A CLASS OF SEMILINEAR ELLIPTIC PROBLEMS

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ON PAIRS OF POSITIVE SOLUTIONS FOR A CLASS OF SEMILINEAR ELLIPTIC PROBLEMS

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ABSTRACT

In this paper ~~we~~ the author discuss the Dirichlet problem

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(1) $-\Delta u = f(u)$ in Ω, $u > 0$ in Ω, $u = 0$ on $\partial\Omega$

infinity

under the hypotheses of sublinearity at 0 and superlinearity at $+\infty$. The dominating theme throughout the paper is that of a supersolution of (1). ~~We~~ They prove theorems on the existence of two solutions whenever problem (1) possesses a supersolution, using topological degree arguments or variational methods according to the type of growth of f at $+\infty$. infinity ~~We also treat~~ also treated are the questions of existence of supersolutions and their actual construction. Schwarz symmetrization techniques are used to obtain supersolutions from solutions of associated symmetrized problems.

AMS (MOS) Subject Classifications: 35J20, 35J25, 47H15

Key Words: Semilinear elliptic problems, Supersolutions, Mountain Pass theorem, A priori estimates, Palais Smale condition, Schwarz symmetrization.

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SIGNIFICANCE AND EXPLANATION

Boundary value problems of the type

(1) $$-\Delta u = f(u) \quad \text{in} \quad \Omega, \ u > 0 \quad \text{in} \quad \Omega, \ u = 0 \quad \text{on} \quad \partial\Omega \ ,$$

arise in a variety of situations, like nonlinear diffusion generated by nonlinear sources, thermal ignition of gases, quantum field theory, gravitational equilibrium of stars, etc. In many applications the nonlinearity f is sublinear at 0 and superlinear at ∞. This class of problems is discussed here, and existence of two solutions of (1) is proved. The interaction of f with the first eigenvalue of $(-\Delta, H_0^1)$ plays an important role in the analysis. The Schwarz symmetrization is used to establish existence of supersolutions of (1). The supersolutions are shown to yield the existence of at least two solutions via topological degree arguments and variational methods.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the authors of this report.

ON PAIRS OF POSITIVE SOLUTIONS FOR A CLASS OF SEMILINEAR ELLIPTIC PROBLEMS

Djairo G. de Figueiredo* and Pierre-Louis Lions**

INTRODUCTION. The question of existence of positive solutions for semilinear elliptic problems of the type

(1) $$-\Delta u = f(u) \text{ in } \Omega,\ u = 0 \text{ on } \partial\Omega ,$$

depends very strongly on the behavior of the function $f : \mathbb{R}^+ \to \mathbb{R}$ at 0 and at $+\infty$. Here Ω denotes a smooth bounded domain in $\mathbb{R}^N$, $N \geq 2$, and f is always assumed to be locally Lipschitzian. We distinguish two special classes of such problems, which have been extensively studied in recent years: the sublinear problems and the superlinear ones. See, for instance, the review papers [1], [2], [3]. The sublinear problems are characterized by the inequalities

(2) $$\liminf_{s\to 0+} \frac{f(s)}{s} > \lambda_1 \quad \text{and} \quad \limsup_{s\to +\infty} \frac{f(s)}{s} < \lambda_1 .$$

Here $\lambda_1 = \lambda_1(\Omega)$ denotes the first eigenvalue of $-\Delta$ in $H_0^1(\Omega)$. The superlinear problems are characterized by the inequalities

(3) $$\limsup_{s\to 0+} \frac{f(s)}{s} < \lambda_1 \quad \text{and} \quad \liminf_{s\to +\infty} \frac{f(s)}{s} > \lambda_1 .$$

In the present paper we propose to discuss problems which are sublinear at 0 and superlinear at $+\infty$. Namely,

(4) $$\liminf_{s\to 0+} \frac{f(s)}{s} > \lambda_1 \quad \text{and} \quad \liminf_{s\to +\infty} \frac{f(s)}{s} > \lambda_1 .$$

We allow these limits to be $+\infty$. For instance, this is the case for the

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first limit in (4) if $f(0) > 0$. And the second limit in (4) is $+\infty$ if f grows more rapidly than linearly. We remark that these types of problems have been studied in the case of $f(0) > 0$ and f convex by several authors, for instance, Gelfand [4], Joseph-Lundgren [5], Crandall-Rabinowitz [6], Bandle [7], Mignot-Puel [8]. Also some problems of this type are discussed in Lions [2]. In this paper we treat a much larger class of nonlinearities. The dominating theme throughout the paper is that of a supersolution of (1). The main highlights being: (i) the question of existence of supersolutions, (ii) their actual construction, (iii) how they can be obtained from solutions of the symmetrized problem by the use of Schwarz symmetrization techniques, (iv) existence theorems for problem (1) that possesses a supersolution.

In Section 1 we present results on the existence of two positive solutions for problem (1), one of the basic assumptions being the existence of a strict supersolution. Here topological degree techniques and variational methods are used to treat different classes of nonlinearities. Condition (4) puts no restriction on the growth of f at $+\infty$. To obtain the first solution of (1) this is no serious problem. Indeed, if we know that a supersolution w of (1) exists, it follows from (4) that a subsolution $\underline{u}$ can be found, with $\underline{u} \leq w$. And then the monotone iteration method applies to yield a solution of (1), which is in fact the minimal solution. However, if we are interested in deciding whether or not there are more solutions we need some sort of compactness in our problem. More precisely, if we plan on treating the problem by topological degree arguments, a priori bounds on the solutions are in order. If we are to use variational methods, then a Palais-Smale condition is required on the Euler-Lagrange functional associated with (1). In both cases these properties will be consequences of the conditions assumed on f as far as its growth at $+\infty$ is concerned. We discuss these points in Section 1. More general results are obtained in Section 4.

The existence of a supersolution of (1) is connected with how far below the line $\lambda_1 s$ the graph of $f(s)$ goes. We propose to discuss this matter with some detail in Section 2, showing how to actually construct a supersolution in some examples. In Section 3 we use the idea of Schwarz symmetrization to obtain statements about the existence of a supersolution of (1) from the knowledge of a solution of (1) in the case of a ball.

In Section 4 we invoke some results of [9] and prove a fairly general theorem on the existence of pairs of positive solutions. In this way we are able to treat a class of nonlinearities f more general than those of Section 1. Under these less restrictive assumptions on f it is not clear that one has a priori bounds on the solutions of (1). Also the corresponding Euler-Lagrange functional apparently does not satisfy the Palais-Smale condition. However we are able to proceed via an appropriate truncation of the nonlinearity f.

1. **EXISTENCE THEOREMS**. In this section we establish results on the existence of positive solutions for the Dirichlet problem

$$(1) \qquad -\Delta u = f(u) \quad \text{in} \quad \Omega, \; u = 0 \quad \text{on} \quad \partial\Omega \quad ,$$

where Ω is a smooth bounded domain in R^N, $N \geq 2$, and the nonlinearity f satisfies different sets of conditions, as specified in the sequel. The following conditions will be always assumed:

(f1) $f : R^+ \to R$ is locally Lipschitzian,

(f2) $\liminf\limits_{s \to 0+} \dfrac{f(s)}{s} > \lambda_1$

(f3) $\liminf\limits_{s \to +\infty} \dfrac{f(s)}{s} > \lambda_1$

(f4) $\lim\limits_{s \to +\infty} \dfrac{f(s)}{s^\sigma} = 0$, with $1 < \sigma \leq (N+2)/(N-2)$, if $N \geq 3$
or $1 < \sigma < \infty$, if $N = 2$.

We remark however that condition (f4) *is not needed* when we want just to state the existence of <u>one</u> solution. Its importance appears when we have to prove the existence of <u>two</u> solutions, since in this case a priori bounds on the solutions or a Palais Smale condition is needed. Besides the above stated conditions on f, the following ones will be useful in treating the problem by topological degree arguments. The conditions next have to be assumed <u>only</u> in the case $N \geq 3$:

(f5) $\limsup\limits_{s \to +\infty} \dfrac{sf(s) - \theta F(s)}{s^2 f(s)^{2/N}} \leq 0$, with $\theta \in [0, 2N/(N-2))$, $F(s) = \int_0^s f$

(f6) Ω is convex or $f(s)s^{-p}$ is nonincreasing for $s > 0$, with $p = (N+2)/(N-2)$.

Assumptions (f4), (f5) and (f6) are used in Theorem 1 below only to assert that there exist a priori bounds on the positive solutions of (1), or more generally, a priori bounds on positive solutions of a certain parametrized family of Dirichlet problems. And for that matter we rely on the results of de Figueiredo-Lions-Nussbaum [9]. So Theorem 1 is true if (f4), (f5) and (f6) are replaced by another set of conditions that insures the existence of such bounds. We remark however that these assumptions are not too restrictive. For instance, (f5) is satisfied if σ in (f4) is $< N/(N-2)$ or if $f(s) = cs^p$, with $c > 0$ and $1 < p < (N+2)/(N-2)$. It has been established in [9] that more general conditions on Ω and f also yield to a priori bounds on the positive solutions of (1). Therefore we see that large classes of nonlinearities may be allowed. For the question of a priori bounds on positive solutions of (1) we refer also to the work of Brézis-Turner [10] and Gidas-Spruck [11].

A function $w \in C^{2,\alpha}(\bar{\Omega})$ is said to be a <u>supersolution</u> of (1) if $-\Delta w \geq f(w)$ in Ω and $w \geq 0$ on $\partial\Omega$. A supersolution which is not a solution is said to be strict. Now we state our first existence result.

<u>Theorem 1</u>. <u>In addition to</u> (f1) - (f6), <u>assume that</u> (1) <u>possesses a strict supersolution</u> w. <u>Then</u> (1) <u>has two solutions</u> $0 < u_1 < u_2$, <u>and</u> u_1 <u>is the minimal solution</u>.

<u>Remarks</u>. 1) Conditions (f1) - (f6) are not sufficient to insure the existence of a positive solution of (1). Indeed, take any f such that $f(s) \geq \alpha s$, for $s \geq 0$ and $\alpha > \lambda_1$. Clearly 0 is the only possible solution of (1). Hence we see that the graph of $f(s)$ has to cross the straight line $\lambda_1 s$. And the question is to determine "how much" it has to be below $\lambda_1 s$, so that a positive solution of (1) exists. It is well known that if $f(s) = 0$ for some $s > 0$ then a positive solution of (1) exists under hypotheses (f1) and (f2). We shall see that the graph of f does not have to go below that far. So the interesting case is $f(s) > 0$, for all $s > 0$.

2) The limits in (f2) and (f3) can be $+\infty$. This will be the case, for instance, if $f(0) > 0$ and $f(s)$ behaves like cs^p for large s, with $c > 0$ and $p > 1$.

3) The requirement that the supersolution be strict is essential, as we show next. Let $g : R^+ \to R^+$ be a continuous strictly convex function, with $g(0) > 0$, and consider the eigenvalue problem:

$(2)_\lambda$ $$-\Delta u = \lambda g(u) \text{ in } \Omega, \ u = 0 \text{ on } \partial\Omega .$$

It is known (see for instance, Crandall-Rabinowitz [6]) that there exists $0 < \lambda^* < \infty$ such that $(2)_\lambda$ has a minimal solution u_λ for $0 < \lambda < \lambda^*$, and no solution for $\lambda > \lambda^*$. We recall that this branch of minimal solutions is obtained by a continuation argument using the implicit function theorem. Moreover if g is such that one can prove an a priori bound on this branch of minimal solutions for $0 < \lambda < \lambda^*$, then such a branch actually reaches $\lambda = \lambda^*$. That is, $(2)_\lambda$ has also a minimal solution u_{λ^*} for $\lambda = \lambda^*$. One can in fact prove that such a u_{λ^*} is the unique solution of $(2)_{\lambda^*}$. Now we claim that u_{λ^*} is the only supersolution of $(2)_{\lambda^*}$. Indeed, suppose that v is a strict supersolution of $(2)_{\lambda^*}$. Then $v > u_{\lambda^*}$ and we have

$$-\Delta(v-u_{\lambda^*}) \geq \lambda^*(g(v) - g(u_{\lambda^*})) \geq \lambda^* g'(u_{\lambda^*})(v-u_{\lambda^*})$$

where one of the inequalities above is strict in a set of positive measure, as a consequence of $v \not\equiv u_{\lambda^*}$ and the strict convexity of g. Consequently, the first eigenvalue of the eigenvalue problem

$$[-\Delta - \lambda^* g'(u_{\lambda^*})]u = \mu u \text{ in } \Omega, \ u = 0 \text{ on } \partial\Omega$$

is positive. However, this implies, via the continuation argument, that there exists $\lambda > \lambda^*$ such that $(2)_\lambda$ has a solution. This contradicts the maximality of λ^*.

Proof of Theorem 1. Let us denote by X the Banach space of C^1-functions in $\bar{\Omega}$ which are 0 on $\partial\Omega$, endowed with the usual C^1-norm. Define f for negative s as $f(s) = f(0)$. The solutions of (1) with this extended f are the same as the solutions of the original problem (1). For in either case the solutions are all positive. Let us define the one-parameter family of functions

$$f_\mu(s) = \mu f(s) + (1-\mu)(\lambda s^+ + 1),\ 0 < \mu < 1 \tag{3}$$

where λ is some fixed number $> \lambda_1$ and s^+ denotes the function which is s for $s > 0$ and 0 for $s < 0$. In view of assumptions (f1), (f3), (f4), (f5), (f6), there is a constant $c_1 > 0$ such that

$$\|u\|_X < c_1 \tag{4}$$

for all eventual solutions of the problems

$$-\Delta u = f_\mu(u) \ \text{ in } \Omega,\ u = 0 \ \text{ on } \partial\Omega,\ 0 < \mu < 1 \ . \tag*{$(5)_\mu$}$$

Now let $k > 0$ be such that $f(s) + ks$ is increasing for $s \in [0, \|w\|_{L^\infty}]$. Let $K_\mu = (-\Delta+k)^{-1}(f_\mu(\cdot)+k\cdot)$. More precisely, let us define $K_\mu : X \to X$ as follows $K_\mu v = u$ where u is the solution of the Dirichlet problem

$$-\Delta u + ku = f_\mu(v) + kv \ \text{ in } \Omega,\ u = 0 \ \text{ on } \partial\Omega \ . \tag{6}$$

The mapping K_μ so defined is compact. From the Schauder estimates it follows that there is a constant $c_2 > 0$ such that

$$\|K_\mu v\|_X < c_2,\ \forall\, v \in X,\ 0 < v < w \ . \tag{7}$$

As a consequence of (f2) we see that there is $\varepsilon > 0$ such that $\varepsilon\phi_1$ is a subsolution for <u>all</u> problems $(5)_\mu$. Here ϕ_1 denotes a positive eigenfunction corresponding to $\lambda_1(\Omega)$. Also we may take $\varepsilon\phi_1 < w$ in Ω. It follows from the maximum principle that any solution u of $(5)_\mu$ such that $u > \varepsilon\phi_1$ in Ω <u>is indeed</u> $u > \varepsilon\phi_1$ in Ω and $\frac{\partial u}{\partial\nu} < \varepsilon\frac{\partial\phi_1}{\partial\nu}$ on $\partial\Omega$. Now consider the bounded open set

$$\theta = \{u \in X : \|u\|_X < c_1 + c_2 + 1;\ u > \varepsilon\phi_1 \ \text{ in } \Omega;\ \frac{\partial u}{\partial\nu} < \varepsilon\frac{\partial\phi_1}{\partial\nu} \ \text{ on } \partial\Omega\} \ ,$$

where c_1 and c_2 are the constants defined in (4) and (7), respectively, and $\varepsilon\phi_1$ is the subsolution said above. By the foregoing remarks it follows that $0 \notin (I-K_\mu)(\partial\theta)$. So the degree $d(I-K_\mu,\theta,0)$ is independent of $\mu \in [0,1]$. Clearly the degree $d(I-K_0,\theta,0) = 0$ since $(5)_0$ has no solution.

Hence

$$d(I-K_1,\theta,0) = 0 \quad .$$

Now let us consider the following open subset of θ:

$$\theta' = \{u \in \theta : u < w \ \text{ in } \ \Omega;\ \frac{\partial u}{\partial \nu} > \frac{\partial w}{\partial \nu} \ \text{ on } \ \partial\Omega\} \quad ,$$

and we claim that $d(I-K_1,\theta',0) = 1$. Once this is proved, it follows that $d(I-K_1,\theta\setminus\overline{\theta'},0) = -1$. So $(5)_1$, which is the same as (1), has two solutions $\tilde{u}_1 \in \theta'$ and $u_2 \in \theta\setminus\overline{\theta'}$, which are not necessarily ordered. So we have to proceed further in order to complete the proof of the theorem. Let $v = \min(w,u_2)$. It follows that v is in $W_0^{1,\infty}(\Omega)$, $\varepsilon\phi_1 < v$ and

$$-\Delta v > f(v), \ \text{ in } \ \mathcal{D}'(\Omega) \quad .$$

So the iteration monotone method yields the existence of a solution u of (1), with $0 < u_1 < u_2$ and u_1 is the minimal solution of (1). To finish the proof we have to prove the above claim. To do that observe that K_1 maps θ' into θ'. Let $u_0 \in \theta'$ and consider the constant mapping $C : \theta' \to \theta'$ defined by $C(u) = u_0$. By the convexity of θ', it follows that $I - K_1$ is homotopic to $I - C$ in θ' and $d(I-K_1,\theta',0) = d(I-C,\theta',0)$. But this last degree is trivially equal to 1. The proof is complete. $\square$

Our next result provides existence of two positive solutions for (1) under a different set of assumptions on the nonlinearity f. We shall drop conditions (f5) and (f6), and replace them by (f7) below. Hence a priori bounds on the solutions are not available any longer. Condition (f7) is needed only when $N > 3$ and the σ in (f4) is $> (N+1)/(N-1)$.

(f7) $\displaystyle \liminf_{s\to+\infty} \frac{sf(s)-\theta F(s)}{s^2 f(s)^{2/(N+1)}} > 0$, with $\theta > 2$ and $F(s) = \int_0^s f$.

Theorem 2. In addition to (f1) - (f4) and (f7), assume that problem (1) possesses a strict supersolution w. Then (1) has two positive solutions.

Remark. Assumptions (f4) and (f7) are used to assure that the Euler-Lagrange functional associated with (1) satisfies the Palais-Smale condition. At this point we rely on the work in [12]. We remark that (f7) is satisfied for instance if one assumes the condition introduced by Ambrosetti and Rabinowitz in [13]. Namely, that there are numbers $\theta > 2$ and $s_0 > 0$ such that $\theta F(s) \le sf(s)$ for $s \ge s_0$. So functions f behaving at $+\infty$ like cs^p, with $c > 0$ and $1 < p < (N+2)/(N-2)$ do satisfy (f7). Hence we see that there are classes of functions which satisfy both (f5) and (f7). However it is apparent that there are functions satisfying one of these conditions but not the other.

Proof. Define f for negative s as $f(s) = f(0)$. Let us consider the functional $\Phi : H_0^1 \to R$ defined by

$$\Phi(u) = \frac{1}{2}\int|\nabla u|^2 - F(u), \qquad F(s) = \int_0^s f \quad .$$

In view of hypotheses (f1), (f3), (f4) and (f7), this functional is of class C^1 and satisfies the Palais-Smale condition. The critical points of Φ are precisely the H_0^1-solutions of (1). In view of (f4) these solutions in fact belong to $C^{2,\alpha}(\bar{\Omega})$. This follows readily by a bootstrap argument in the case when σ in (f4) is $< (N+2)/(N-2)$. The case when $\sigma = (N+2)/(N-2)$ is treated by more sophisticated arguments due to Brézis-Kato [14], see also [12]. Now we use the fact that (1) possesses an ordered pair of a sub- and a supersolution: $\varepsilon\phi_1 < w$ for $\varepsilon > 0$ sufficiently small. Then it is known (see [15]) that there is a point u_1 in the interval $[\varepsilon\phi_1, w]$, which is a local minimum of Φ. Now, either (i) u_1 is a strict local minimum of Φ, or (ii) it is not. Hence at this point the proof bifurcates to cover separately these two possibilities. Let us first assume (i). Define the function $g : \Omega \times R \to R$ by $g(x,s) = f(s^+ + u_1(x)) - f(u_1(x))$. The functional $\Psi : H_0^1 \to R$ defined by

$$\Psi(v) = \frac{1}{2}\int |\nabla v|^2 - \int G(x,v) \quad ,$$

where $G(x,s) = F(s^+ + u_1(x)) - f(u_1(x))s^+ - F(u_1(x))$, satisfies the Palais-Smale condition. Its critical points are the H_0^1-solutions of

$$-\Delta v = f(v^+ + u_1) - f(u_1) \quad ,$$

and it follows that they are ≥ 0 a.e. (Ω). So a non-zero critical point v of Ψ provides a second nontrivial solution of problem (1): $u_2 = v + u_1$. To show that Ψ has a non-zero critical point it is enough to prove that 0 is a strict local minimum of Ψ. (The Mountain Pass Theorem then completes the proof). For that matter we observe that there is $\varepsilon > 0$ such that

(*) $$\Psi(v) > 0 \quad \text{for all} \quad 0 < \|v\|_{H^1} < \varepsilon \quad .$$

This is a simple consequence of the identity:

$$\Psi(v) = \frac{1}{2}\int |\nabla v^-|^2 + \Phi(v^+ + u_1) - \Phi(u_1) \quad ,$$

and the fact that u_1 is a strict local minimum of Φ. Next we claim that there is $\alpha > 0$ such that $\Psi(v) \geq \alpha$ for $\|v\|_{H^1} = \varepsilon$. In fact, suppose that this is not the case. Then there exists v_n, with $\|v_n\|_{H^1} = \varepsilon$, and $\Psi(v_n) \to 0$. Since $v_n \rightharpoonup v$ we get $\Psi(v) \leq 0$. In view of (*) above we see that $v = 0$. So passing to a subsequence we see that $\int G(x,v_n) \to 0$, which implies that $\Psi(v_n) \geq \frac{1}{4}\varepsilon^2$ for n sufficiently large. But this contradicts the fact that $\Psi(v_n) \to 0$.

Now we proceed to consider alternative (ii). That is, suppose that given $\varepsilon_0 > 0$ such that $\Phi(u) \geq \Phi(u_1)$ for $\|u-u_1\| < \varepsilon_0$, there is an $0 < \varepsilon < \varepsilon_0$ such that

$$\inf\{\Phi(u) : \|u-u_1\|_{H^1} = \varepsilon\} = \Phi(u_1) \quad .$$

Let (ω_n) be such that $\|\omega_n - u_1\|_{H^1} = \varepsilon$, $\Phi(\omega_n) \to \Phi(u_1)$ and $\omega_n \rightharpoonup \omega$. It follows from (f4) that Φ is weakly lower semicontinuous. So $\Phi(\omega_n) \to \Phi(\omega)$, which gives $\Phi(\omega) = \Phi(u_1)$ showing that ω is a minimum of Φ in the ball

$\|u-u_1\|_{H^1} < \varepsilon_0$. We see readily that $\omega \neq u_1$. Otherwise, supposing $\omega = u_1$, we come to a contradiction passing to th limit in the identity below:

$$\Phi(\omega_n) = \frac{1}{2}\int |\nabla(\omega_n - u_1)|^2 + \int \nabla\omega_n \cdot \nabla u_1 - \frac{1}{2}\int |\nabla u_1|^2 - \int F(\omega_n) \quad .$$

(Recall that the first term in the right side of the previous equality is $\varepsilon^2/2$.) Consequently ω is another critical point of Φ, since $0 < \|\omega - u_1\| < \varepsilon < \varepsilon_0$. $\square$

<u>Remark</u>. If $f(0) > 0$ the above proof can be considerably shortened. Indeed, at the point where it bifurcates we have only to apply the strong form of the Mountain Pass Theorem, as proved by Rabinowitz [16]; see also a simple proof in [15]. The critical point so obtained is clearly nontrivial. We also remark that the above proof in the case of alternative (ii) can also be done using Ekeland variational principle as in [15].

2. <u>The case when Ω is a ball</u>. Suppose that Ω is the open ball $B_\rho = \{x \in R^N : \|x\| < \rho\}$. In this section we show how to construct supersolutions of problem (1), and discuss some examples. Let us begin by considering the following linear Dirichlet problem:

(8) $$-\Delta v = \mu v + 1 \quad \text{in } B_1,\ v = 0 \quad \text{on } \partial B_1 \ ,$$

where $0 < \mu < \lambda_1(B_1)$. Let us denote by v_μ the solution of (8). We know that v_μ is positive, radially symmetric and its maximum M_μ is assumed at the center of the ball. That is

$$M_\mu = v_\mu(0) = \max\{v_\mu(x) : x \in B_1\} \ .$$

These functions v_μ and their corresponding maxima, M_μ, play an important role in the questions of existence of solutions for semilinear problems like (1). Problem (8) can be solved explicitely. Since $v(x)$ is a function only of the radius $r = |x|$, (denote it again by $v(r)$), it has to be the solution of the following two-point boundary value problem

(9) $$-v'' - \frac{N-1}{r} v' = \mu v + 1 \quad \text{in } (0,1),\ v'(0) = v(1) = 0 \ .$$

In the case $N = 2$ this equation becomes

$$v'' + \frac{1}{r} v' + \mu v = -1$$

which is the Bessel's equation of zero order. So the solution of (9) is

$$v_\mu(r) = \frac{1}{\mu} \left[\frac{J_0(\sqrt{\mu}\, r)}{J_0(\sqrt{\mu})} - 1\right]$$

where J_0 denotes the Bessel function of zero order. Recalling that $J(0) = 1$, we obtain a simple expression M_μ in this case. Namely

$$M_\mu = \frac{1}{\mu} \left[\frac{1}{J_0(\sqrt{\mu})} - 1\right] \ .$$

In the case $N \geq 3$, we make the substitution $\omega(r) = v(r) r^p$, where $p = (N-2)/2$. So ω is a solution of

$$\omega'' + \frac{1}{r} \omega' + (\mu - \frac{p^2}{r^2})\omega = -r^p,\ \omega(0) = \omega(1) = 0 \ ,$$

which is given readily in terms of Bessel's functions of order p. Thus the solution of (9) is

$$v_\mu(r) = \frac{1}{\mu}\left[\frac{J_p(\sqrt{\mu}\, r)}{r^p J_p(\sqrt{\mu})} - 1\right], \qquad p = (N-2)/2 \ .$$

Recalling that $r^{-p}J_p(r) \to (2^p\Gamma(p+1))^{-1}$ as $r \to 0$, we obtain

$$M_\mu = \frac{1}{\mu}\left[\frac{\mu^{p/2}}{2^p\Gamma(p+1)J_p(\sqrt{\mu})} - 1\right], \qquad p = (N-2)/2 \ ,$$

which of course is valid for all $N > 2$. In the case $N = 3$, the expressions of v_μ and M_μ simplify to more elementary functions. Namely,

$$v_\mu(r) = \frac{1}{\mu}\left[\frac{\sin(\sqrt{\mu}\, r)}{r \sin\sqrt{\mu}} - 1\right], \qquad M_\mu = \frac{1}{\mu}\left[\frac{\sqrt{\mu}}{\sin\sqrt{\mu}} - 1\right] \ .$$

From the former expressions we see the following properties of M_μ which will be useful in the sequel. Recalling that $r^{-2}(r^{-p}J_p(r)2^p\Gamma(p+1) - 1] \to [2(2p+2)]^{-1}$ as $r \to 0$, we conclude that

$$M_0 = (2N)^{-1} \ .$$

It is easily seen that the first eigenvalue $\lambda_1(B_1)$ of $-\Delta\phi = \lambda\phi$ in B_1, $\phi = 0$ on ∂B_1 is the square of the first positive zero ν_p of $J_p(r)$: $\lambda_1(B_1) = \nu_p^2$. Consequently $M_\mu \to +\infty$ as $\mu \to \lambda_1(B_1)$. Since $M_\mu < M_{\mu'}$ for $0 < \mu < \mu' < \lambda_1(B_1)$ we have a pretty good idea of the graph of M_μ.

Proposition 3. Suppose that the function $f : R^+ \to R^+$ satisfies the following condition: there are numbers $s_0 > 0$ and $0 < \mu < \lambda_1(B_1)$ such that

(f8) $f(s) \le \mu s + M_\mu^{-1} s_0, \ \forall\, 0 \le s \le s_0$.

Then problem (1) with $\Omega = B_1$ has a strict supersolution.

Remark. A special case of (f8) is

$(f8)_0$ $f(s) \le 2Ns_0, \ \forall\, 0 \le s \le s_0$, for some $s_0 > 0$.

Proof. The solution w of the linear Dirichlet problem

$$(8') \qquad -\Delta w = \mu w + C \text{ in } B_1,\ w = 0 \text{ on } \partial B_1$$

is such that $0 \le w \le M_\mu C$, for any given constant $C > 0$. We see that w is a supersolution of (1) provided $\mu s + C \ge f(s)$ for all $0 \le s \le M_\mu C$. But this follows from (f8) taking $C = M_\mu^{-1} s_0$. □

Remark. In the case when $\Omega = B_\rho$, the analogue of condition (f8) is: there are numbers $s_0 > 0$ and $0 \le \mu < \lambda_1(B_\rho)$ such that

$$f(s) \le \mu s + \rho^{-2} M_{\mu\rho^2}^{-1} s_0,\ \forall\, 0 \le s \le s_0 \ .$$

Remark. The idea of using linear Dirichlet problems like (8') to obtain a supersolution of (1) has been used before by several authors. A special case of Proposition 3 for increasing functions f was obtained by Bandle [23] using this method.

Examples. 1) $f(s) = \lambda e^s$. Consider the nonlinear eigenvalue problem

$$(10) \qquad -\Delta u = \lambda e^u \text{ in } B_1,\ u = 0 \text{ on } \partial B_1 \ .$$

It is known that there is a $\bar{\lambda} > 0$ such that (10) has solutions for $\lambda \le \bar{\lambda}$ and no solution if $\lambda > \bar{\lambda}$. What can be said about the value of $\bar{\lambda}$? In the case when $N = 2$, problem (10) can be solved explicitly. This was already proved by Liouville in 1853, see Bandle [17]. In this case it is seen that $\bar{\lambda} = 2$. Proposition 3 above provides lower bounds for $\bar{\lambda}$ in any dimension N. For instance, $(f8)_0$ in the case of $N = 2$, gives $\bar{\lambda} \ge 4/e = 1.47$. Using (f8) we can improve this bound and get

$$(11) \qquad \bar{\lambda} \ge \max\left\{\frac{\ln(1+\mu M_\mu)}{M_\mu} : 0 \le \mu < \lambda_1(B_1)\right\}$$

which in the case of $N = 2$ gives $\bar{\lambda} \ge 1.8043$. In the case of $N = 3$, $(f8)_0$ gives $\bar{\lambda} \ge 6/e = 2,21$, while estimate (11) gives $\bar{\lambda} \ge 2.8652$.

2) $f(s) = \lambda(1+\alpha s)^\beta$, $\alpha, \beta > 0$. We see then that the problem

$$-\Delta u = \lambda(1+\alpha u)^\beta \text{ in } B_1,\ u = 0 \text{ on } \partial B \ ,$$

has a solution for each $\lambda \le \tilde{\lambda}$ where

$$\tilde{\lambda} = \max\{\frac{1}{\alpha M_\mu}[(1+\mu M_\mu)^{1/\beta} - 1] : 0 < \mu < \lambda_1(B_1)\} \quad .$$

3. The case of a general domain Ω. In this section we use Schwarz symmetrization for two purposes, First, we state a condition on f (depending only on the volume of Ω) which insures the existence of a strict supersolution of (1) for a general domain Ω. Second, we prove that the existence of a solution of (1) for the ball with same volume as Ω implies the existence of a solution for general Ω. This result is useful in constructing strict supersolutions of (1) as we shall see.

Let us begin by reviewing the basic facts about symmetrization, which will be used in the sequel. More about symmetrization can be seen in Hardy-Littlewood-Pólya [18], Pólya-Szegö [19] and Bandle [17]. Let $u : R^N \to R$ be an L^1-function. We define the Schwarz symmetrization u^* (also called the symmetric decreasing rearrangement) of u as the function $u^* : R^N \to R$ such that

$$\text{meas}\{x \in R^N : |u(x)| > t\} = \text{meas}\{x \in R^N : u^*(x) > t\}, \qquad \forall\, t > 0 \quad .$$

It follows that u^* is radially symmetric, positive and nonincreasing with respect to $|x|$. And one sees that

$$u^*(x) = \inf[t > 0 : \text{meas}\{x \in R^N : |u(x)| > t\} < \omega_N |x|^N] \quad ,$$

where ω_N is the volume of the unit ball in R^N.

The following properties of u^* will be needed here

(I) If $u \in L^p(R^N)$, $1 < p < \infty$, then $u^* \in L^p(R^N)$ and $\|u\|_{L^p} = \|u^*\|_{L^p}$.

(II) If $u \in H^1(R^N)$, then $u^* \in H^1(R^N)$, and $\|\nabla u^*\|_{L^2} < \|\nabla u\|_{L^2}$.

The proof of (I) is given in [18] or [19]. An elementary proof of (II) can be seen in [20]. Now let Ω be a smooth bounded domain in R^N. Let us denote by Ω^* the ball centered at 0 and with the same volume as Ω. Then, it follows from (I) and (II) that

(III) $\lambda_1(\Omega^*) < \lambda_1(\Omega)$.

The next result is proved in Lions [21]:

(IV) Consider the following two linear Dirichlet problems:

$$-\Delta u = \lambda u + 1 \ \text{ in } \ \Omega, \ u = 0 \ \text{ on } \ \partial\Omega \ ;$$

$$-\Delta v = \lambda v + 1 \ \text{ in } \ \Omega^*, \ v = 0 \ \text{ on } \ \partial\Omega^*$$

where $0 < \lambda < \lambda_1(\Omega^*)$. Then $u^* < v$ a.e. (Ω^*).

(V) If u is a Lipschitz continuous function in Ω, then u^* is Lipschitz continuous in Ω^*. See [17]. The next result is due to Talenti [22]. A simple proof can be seen in [21]. We remark that in fact a more general result is true, as proved in both [21] and [22].

(VI) Let h be a Lipschitz continuous function in Ω. Consider the following two Dirichlet problems:

$$-\Delta u = h \ \text{ in } \ \Omega, \ u = 0 \ \text{ on } \ \partial\Omega \ ;$$

$$-\Delta v = h^* \ \text{ in } \ \Omega^*, \ v = 0 \ \text{ on } \ \partial\Omega^* \ .$$

Then $u^* < v$ a.e. (Ω^*).

Now we give a simple sufficient condition on f for the existence of a strict supersolution of (1).

Proposition 4. *Consider* (1) *for a smooth bounded domain* Ω *in* R^N, *and let* Ω^* *be the ball in* R^N *with the same volume as* Ω. *Suppose that the function* f *satisfies the following condition: there exist numbers* $s_0 > 0$ *and* $0 < \mu < \lambda_1(\Omega^*)$ *such that*

$$f(s) < \mu s + \rho^{-2} M^{-1}_{\mu\rho^2} s_0, \ \forall \ 0 < s < s_0 \ ,$$

where ρ *is the radius of the ball* Ω^*. *Then the problem* (1) *has a strict supersolution.*

Proof. Proceed as in the proof of Proposition 3, using now (I) and (IV) above. □

Now let us show how a solution of (1) can be obtained from a solution of

(12) $\qquad -\Delta v = f(v) \;\text{ in }\; \Omega^*,\; v > 0 \;\text{ in }\; \Omega^*,\; v = 0 \;\text{ on }\; \partial\Omega^*$.

Theorem 5. *Let* $f : R^+ \to R^+$ *be a nondecreasing function satisfying hypotheses* (f1) *and* (f2). *Suppose that* (12) *has a solution* v. *Then problem* (1) *has a minimal solution* $\underline{u}$ *and* $\underline{u}^* < \underline{v}$, *where* $\underline{v}$ *is the minimal solution of* (12).

Proof. Let us denote by $\hat{\phi}_1$ the eigenfunction corresponding to $\lambda_1(\Omega^*)$, which is > 0 and $\int \hat{\phi}_1^2 = 1$. Choose $\varepsilon > 0$ and $\delta > 0$ such that $u_0 = \varepsilon\phi_1$ is a subsolution of (1), $v_0 = \delta\hat{\phi}_1$ is a subsolution of (12) and

$$(13)_0 \qquad u_0^* < v_0 \;.$$

Now we define iteratively the sequences (u_n), (v_n) by

$$-\Delta u_{n+1} = f(u_n) \;\text{ in }\; \Omega \qquad u_{n+1} = 0 \;\text{ on }\; \partial\Omega \;,$$

$$-\Delta v_{n+1} = f(v_n) \;\text{ in }\; \Omega^* \qquad v_{n+1} = 0 \;\text{ on }\; \partial\Omega^* \;.$$

In view of (VI), $u_1^* < w_1$, where w_1 is the solution of

$$-\Delta w_1 = f(u_0^*) \;\text{ in }\; \Omega^* \;,\; w_1 = 0 \;\text{ on }\; \partial\Omega^* \;.$$

So $-\Delta w_1 < f(v_0) = -\Delta v_1$, which implies by the maximum principle $v_1 > w_1$. Consequently

$$(13)_1 \qquad u_1^* < v_1 \;.$$

By induction, it follows that

$$(13)_n \qquad u_n^* < v_n \;.$$

But, since problem (12) has a solution, it follows that (v_n) converges to $\underline{v}$ in the $C^{2,\alpha}$-norm. Now from (I) it follows that $\|u_n\|_{L^\infty} = \|u_n^*\|_{L^\infty} < \|v_n\|_{L^\infty}$. So (u_n) also converges to a function $\underline{u}$ which is the minimal solution of (1). And from (13) we obtain $\underline{u}^* < \underline{v}$.

The following results show how Theorem 5 can be used to establish the existence of strict supersolutions of (1).

Corollary 6. *Let* $f_1 : R^+ \to R^+$ *be an increasing function satisfying hypotheses* (f1) *and* (f2), *such that* $f_1(s) > f(s)$ *for* s *in some interval* $(0,\varepsilon)$, *and* $f_1(s) > f(s)$ *for all* $s \in R^+$. *Suppose that the problem*

$$-\Delta v = f_1(v) \quad \text{in} \quad \Omega^*, \; v = 0 \quad \text{on} \quad \partial\Omega^*$$

has a solution. Then (1) *possesses a strict supersolution* w.

Proof. Indeed, w is the solution of the problem

$$-\Delta w = f_1(w) \quad \text{in} \quad \Omega, \; w = 0 \quad \text{on} \quad \partial\Omega$$

given by Theorem 5. □

Corollary 7. Let f *satisfy all the hypotheses of Theorem 5. Moreover* f *is* C^1. *Suppose that the first eigenvalue of* $-\Delta\omega - f'(v)\omega = \lambda\omega$ *in* Ω^*, $\omega = 0$ *on* $\partial\Omega^*$ *is positive. Then problem* (1) *has a strict supersolution.*

Proof. By the implicit function theorem the problem

$$-\Delta z = f(z) + \varepsilon z \quad \text{in} \quad \Omega^*, \; z = 0 \quad \text{on} \quad \partial\Omega^*$$

has a solution z. So by Theorem 5, the analogous problem in Ω has also a solution, u. Clearly u is a strict supersolution of (1). □

We have seen above that the existence of a solution in the ball Ω^* implies the existence of a minimal positive solution in Ω. In fact, using the full generality of [22] - see also [21] -, we may improve this argument as follows. Assume that f is nondecreasing, (f1) holds and that there exists v solution of:

$$(12) \qquad -\Delta v = f(v) \quad \text{in} \quad \Omega^*, \; v > 0 \quad \text{in} \quad \Omega^*, \; v = 0 \quad \text{on} \quad \partial\Omega^* \; .$$

Then let $a_{ij}(x) = a_{ji}(x) \in L^\infty(\Omega)$ satisfy: $(a_{ij}(x)) \geq I_N$ a.e. in Ω and consider the following problem:

$$-\frac{\partial}{\partial x_i} a_{ij}(x) \frac{\partial u}{\partial x_j} = f(u) \quad \text{in} \quad \Omega, \; u > 0 \quad \text{in} \quad \Omega, \; u = 0 \quad \text{on} \quad \partial\Omega \; .$$

If $\liminf\limits_{t \to 0+} \frac{f(t)}{t} > \lambda_1(-\frac{\partial}{\partial x_i}(a_{ij}\frac{\partial}{\partial x_j}), H_0^1(\Omega))$, then the preceding problem has a minimal solution u such that:

$$u^* \leq \underline{v} \; ,$$

where $\underline{v}$ is the minimal solution of (12).

4. <u>EXISTENCE RESULTS WITH NEITHER A PRIORI BOUNDS NOR PALAIS-SMALE</u>. In this section we consider once more problem (1) under the basic assumption that a strict supersolution exists. Again conditions (f1) through (f4) are assumed. However we shall assume neither condition (f5) nor (f7). Consequently one does not know whether the solutions of (1) are a priori bounded. Also the corresponding Euler-Lagrange functional does not clearly satisfy the Palais-Smale condition. Thus the methods of Section 1 do not apply directly. We found it useful to rely on the results of [9]. It is proved there that the bound

(14) $$\|\nabla u\|_{L^\infty(\partial\Omega)} < k_0 \quad , \quad \text{for all solutions of (1)} \quad ,$$

holds if in addition to (f1), (f3) and (f4) one assumes condition (f6). Also we recall (c.f. [9]) that the condition below in conjunction with (f1), (f3) and (f4) gives inequality (17) used in the proof of Theorem 8 presented next.

(f9) $\partial\Omega = \Gamma_1 \cup \Gamma_2$, where Γ_1 and Γ_2 are closed and satisfy

(i) at every point of Γ_1 all sectional curvatures of Γ_1 are bounded away from 0 by a positive constant;

(ii) there is a point x_0 in R^N such that $(x-x_0, n(x)) < 0$ for all $x \in \Gamma_2$, where $n(x)$ denotes the outward unit normal to $\partial\Omega$ at x.

<u>Theorem 8</u>. <u>In addition to (f1) - (f4), assume that (1) possesses a strict supersolution</u> w. <u>Suppose that one of the conditions (f6) or (f9) holds. Then (1) has two solutions</u> $0 < u_1 < u_2$.

<u>Proof</u>. i) We first observe that (f3) implies that there exist $\mu > \lambda_1$ and $R > 0$ such that

(15) $$f(s) > \mu s \quad , \quad \text{for all} \quad s > R \quad .$$

Also it follows from (f4) that given $\varepsilon > 0$ there is $C_\varepsilon > 0$, such that

(16) $$f(s) < \varepsilon s^{(N+2)/(N-2)} + C_\varepsilon \quad , \quad \text{for all} \quad s > 0 \quad .$$

If (f6) is assumed we have from [9] that (14) holds, with k_0 depending only on μ and R given in (15). Using Pohozaev's identity (cf. [9]) we see that there exists a constant $k_1 > 0$ depending only on μ and R, such that

$$\left|\frac{N-2}{2N}\int_\Omega |\nabla u|^2 - \int_\Omega F(u)\right| \le k_1 \quad . \tag{17}$$

If we assume (f9) we also obtain (17) from the results of [9]. So in any case we have that (17) holds true for all solutions of (1).

ii) By the results of Section 1 there is $u_1 \in H_0^1$ in the interval $[0,w]$ which is a local minimum of

$$\Phi(v) = \frac{1}{2}\int_\Omega |\nabla v|^2 - \int_\Omega F(v) \quad .$$

Consider the functional $\psi : H_0^1 \to R$ defined by

$$\psi(v) = \frac{1}{2}\int_\Omega |\nabla v|^2 - \int_\Omega G(x,v)$$

where $G(x,s) = F(s^+ + u_1(x)) - f(u_1(x))s^+ - F(u_1(x))$. Let $C_0 = \|w\|_{L^\infty} + 1$. It follows from (f3) that there is a $v_0 \in C^1(\bar\Omega)$, depending only on C_0, μ and R, such that $v_0 > 0$ in Ω and $\psi(v_0) < 0$. Let

$$C_1 = \|u_1 + v_0\|_{L^\infty} + 1 \quad \text{and} \quad b = \max\{\psi(tv_0) : 0 \le t \le 1\} \quad .$$

Then b depends only on f on the interval $[0,C_2]$ where $C_2 = \max(C_0,C_1)$ and on μ and R.

iii) Now if v is a critical point of ψ, we see that $v > 0$ and $u = v + u_1$ is a solution of (1). If we suppose that $\psi(v) \in [0,b]$, then it follows from (17) that

$$\|u\|_{L^\infty} \le C_3$$

where C_3 depends on k_1, b, C_ε and f restricted to the interval $[0,C_0]$. Consequently C_3 depends only on μ, R, C_ε and f restricted to the interval $[0,C_2]$. Therefore we modify f outside of the interval $[0,\max(C_2,C_3)]$, keeping the same μ, R and C_ε. Moreover this modification is done in such a way that the new f satisfies the assumptions which enable us to use the results of Section 1. Thus the critical point v of the modified ψ obtained by mountain passing satisfies $\psi(v) \in [0,b]$. It is then readily seen that $u = v+u_1$ is the second solution of (1) we are looking for. $\square$

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In this paper we discuss the Dirichlet problem

$$(1) \qquad -\Delta u = f(u) \text{ in } \Omega, \; u > 0 \text{ in } \Omega, \; u = 0 \text{ on } \partial\Omega$$

under the hypotheses of sublinearity at 0 and superlinearity at $+\infty$. The dominating theme throughout the paper is that of a supersolution of (1). We prove theorems on the existence of two solutions whenever problem (1) possesses a supersolution, using topological degree arguments or variational methods according to the type of growth of f at $+\infty$. We also treat the questions of existence of supersolutions and their actual construction. Schwarz symmetrization techniques are used to obtain supersolutions from solutions of associated symmetrized problems.